TEMPERATURE DISTRIBUTION IN LAMINAR FLOW OF AN INCOMPRESSIBLE FLUID IN A RECTANGULAR CHANNEL ALLOWING FOR ENERGY DISSIPATION

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Expressions have been obtained for the temperature distribution over the section and heat flux through the channel wall in the presence of energy dissipation for the case of laminar flow of a liquid in an infinitely long channel of rectangular section with constant wall temperature.

Many physical and chemical investigations necessarily involve laminar flow of a viscous liquid in a channel of rectangular section with constant wall temperature. Even for velocities small in absolute magnitude, but having large gradients due to dissipation of mechanical energy, heating of the liquid occurs, and its temperature may differ considerably from that of the channel walls, especially for viscous liquids. This heating of the liquid must be taken into account in accurate investigations; moreover, during flow, for example, of chemically reacting substances, the reaction rate may depend appreciably on stream temperature, and in that event the liberation of frictional heat may have a substantial influence on the course of the chemical reaction.

Exact solutions exist for the differential equations of motion and energy, allowing for friction heating, for example, that of Schlichting for a plane narrow channel [1].

This paper examines laminar flow of a liquid in a channel of rectangular section, taking account of energy dissipation. Results of an analytical solution are presented, and the temperature distribution over the channel section is obtained.

The differential equations of motion and energy [1] in a rectangular-channel flow, taking account of dissipation of mechanical energy, have the form, respectively,

$$\frac{\partial^2 W_z}{\partial x^2} + \frac{\partial^2 W_z}{\partial y^2} = \frac{1}{\mu}\frac{\partial P}{\partial z}; \tag{1}$$

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = -\frac{\mu}{\lambda}\left[\left(\frac{\partial W_z}{\partial x}\right)^2 + \left(\frac{\partial W_z}{\partial y}\right)^2\right]. \tag{2}$$

Equations (1) and (2) were obtained under the following conditions:

1. Steady established flow of an incompressible liquid with constant physical properties, independent of temperature, in an infinitely long channel.

2. Liquid flow laminar, i.e., the velocity components across the section are zero $W_x = W_y = 0$ and $P = \text{const}$.

3. Influence of body forces (gravity forces) not taken into account.

The solution of (1) and (2) is carried out with the following boundary conditions (Fig. 1):

$$x = 0;\ x = a;\ W_z = 0;\ t = t_c,$$
$$y = 0;\ y = b;\ W_z = 0;\ t = t_c. \tag{3}$$

For simplicity, we examine flow in a channel with identical wall temperature, since a difference in the temperatures of the walls does not introduce a difference in principle into the solution of the problem.

In conformity with the assumed constancy of the physical properties, Eqs. (1) and (2) may be solved independently of one another. Transferring in (1) and (2) and boundary conditions (3) to the new variables

$$X = \frac{x}{a}\pi,\ Y = \frac{y}{b}\pi,\ T = t - t_c,$$

we apply a finite integral Fourier sine transformation with respect to the variable X [2] to solve the equation of motion (1). Allowing for the boundary conditions, the result may be written in the form of the following series:

$$W_z = \frac{4A}{\pi}\sum_{p=1,3,5\ldots}^{\infty}\left[-1 + \operatorname{ch}\left(pY\frac{b}{a} - p\pi\frac{b}{2a}\right)\Big/\operatorname{ch}\frac{p\pi b}{2a}\right]\frac{\sin pX}{p^3}, \tag{4}$$

where

$$A = \frac{a^2}{\pi^2}\frac{1}{\mu}\frac{\partial P}{\partial z}.$$

The velocity distribution presented in [4] looks somewhat different because of a different choice of coordinate origin.

We shall use velocity distribution (4) to solve the energy equation. Substituting the expression for derivatives of velocity with respect to the coordinates in the energy equation, and once more applying the finite integral Fourier sine transformation with respect to the variable X, we obtain, taking account of the boundary conditions,

$$\frac{a^2}{b^2}\frac{\partial^2 \bar{T}}{\partial Y^2} - k^2\bar{T} = -\frac{\mu}{\lambda}\frac{16A^2}{\pi^2}\left\{\left[\left(1 + \frac{\operatorname{ch}^2\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}^2\frac{\pi b}{2a}} - \frac{2\operatorname{ch}\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{\pi b}{2a}}\right)\frac{2(k^2-2)}{k(k^2-4)} + \ldots + \frac{1}{9}\left(1 - \frac{\operatorname{ch}\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{\pi b}{2a}} - \frac{\operatorname{ch}3\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}3\frac{\pi b}{2a}} + \frac{\operatorname{ch}\frac{b}{a}\left(Y-\frac{\pi}{2}\right)\operatorname{ch}3\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{\pi b}{2a}\operatorname{ch}3\frac{\pi b}{2a}}\right) \times \left(\frac{2k}{k^2-4} + \frac{2k}{k^2-16}\right) + \frac{1}{25} \times \left(1 + \frac{\operatorname{ch}\frac{b}{a}\left(Y-\frac{\pi}{2}\right)\operatorname{ch}5\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{\pi b}{2a}\operatorname{ch}5\frac{\pi b}{2a}} - \frac{\operatorname{ch}\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{\pi b}{2a}} - \frac{\operatorname{ch}5\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}5\frac{\pi b}{2a}}\right)\left(\frac{2k}{k^2-16} + \frac{2k}{k^2-36}\right) + \ldots\right] + \left[\frac{\operatorname{sh}^2\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}^2\frac{\pi b}{2a}}\frac{4}{k(4-k^2)} + \ldots + \frac{1}{9} \times \frac{\operatorname{sh}\frac{b}{a}\left(Y-\frac{\pi}{2}\right)\operatorname{sh}3\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{\pi b}{2a}\operatorname{ch}3\frac{\pi b}{2a}} \cdot \left(\frac{2k}{k^2-4} - \frac{2k}{k^2-16}\right) + \frac{1}{25}\frac{\operatorname{sh}\frac{b}{a}\left(Y-\frac{\pi}{2}\right)\operatorname{sh}5\frac{b}{a}\left(Y-\frac{\pi}{2}\right)}{\operatorname{ch}\frac{\pi b}{2a}\operatorname{ch}5\frac{\pi b}{2a}} \times \left(\frac{2k}{k^2-16} - \frac{2k}{k^2-36}\right) + \ldots\right]\right\}, \quad (5)$$

where

$$\bar{T} = \int_0^\pi T\sin kX\,dX,$$

and k is an odd integer.

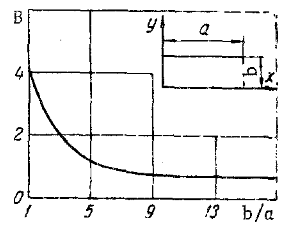

Fig. 1. Heating of the liquid on the channel axis as a function of the ratio of the sides $B \equiv t/(\mu/\lambda)\,\overline{W}_Z^2$.

We write the boundary conditions for $\bar{T}$:

$$Y = 0;\ Y = \pi;\ \bar{T} = 0. \quad (6)$$

The general solution for (5) may be written as follows [3]:

$$\bar{T} = C_1\exp\left(k\frac{b}{a}Y\right) + C_2\exp\left(-k\frac{b}{a}Y\right) + f(Y), \quad (7)$$

where $f(Y)$ is a particular solution of (5).

Determining the constants C_1 and C_2 from boundary conditions (6), we obtain an expression for $\bar{T}$:

$$\bar{T} = \frac{f(0)\left[\exp(-k\pi b/a) - f(\pi)/f(0)\right]}{\exp(k\pi b/a) - \exp(-k\pi b/a)}\exp\left(kY\frac{b}{a}\right) + \frac{f(0)\left[f(\pi)/f(0) - \exp(k\pi b/a)\right]}{\exp(k\pi b/a) - \exp(-k\pi b/a)}\exp\left(-kY\frac{b}{a}\right) + f(Y), \quad (8)$$

where $f(0)$ and $f(\pi)$ are values of $f(Y)$ when $Y = 0$ and $Y = \pi$.

Using the conversion formula [2], we obtain the temperature distribution over the channel section:

$$t = \frac{2}{\pi}\sum_{k=1}^{\infty}\bar{T}\sin kX = \frac{2}{\pi}\sum_{k=1,3,5\ldots}^{\infty}\sin kX\left\{\frac{f(0)}{\exp(k\pi b/a) - \exp(-k\pi b/a)} \times \left[\left(\exp\left(-k\pi\frac{b}{a}\right) - \frac{f(\pi)}{f(0)}\right)\exp\left(kY\frac{b}{a}\right) + \left(-\exp\left(k\pi\frac{b}{a}\right) + \frac{f(\pi)}{f(0)}\right)\exp\left(-kY\frac{b}{a}\right)\right] + f(Y)\right\}. \quad (9)$$

Using the temperature distribution obtained, we may obtain an expression for the heat flux through any wall:

$$q = -\lambda\left(\frac{\partial t}{\partial n}\right)_W. \tag{10}$$

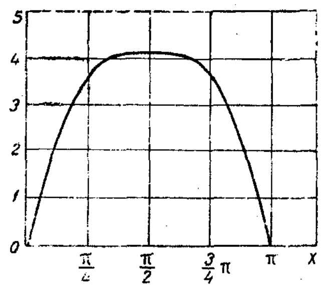

Fig. 2. Temperature distribution in the mid-plane ($Y = \pi/2$) of a square channel $B \equiv t/(\mu/\lambda)\,\overline{W}_Z^2$.

Thus, for the heat flux at the wall when $X = 0$ the following expression is obtained:

$$q = -\frac{2\lambda}{a}\sum_{k=1,3,5\ldots}^{\infty} k\left\{\frac{f(0)}{\exp(k\pi b/a) - \exp(-k\pi b/a)}\times\right.$$
$$\times\left[(\exp(-k\pi b/a) - f(\pi)/f(0))\exp(kY b/a) + \right. \tag{11}$$
$$\left.\left.+\left(-\exp\left(k\pi\frac{b}{a}\right) + \frac{f(\pi)}{f(0)}\right)\exp\left(-kY\frac{b}{a}\right)\right] + f(Y)\right\}.$$

The convergence of the series (9) deteriorates with decrease of the ratio b/a. Thus, in calculations of the temperature on the channel axis using the first three values of the number p (1, 3, 5), we obtained the result that for $b/a = 1$ the value of the third term of series (9) ($k = 5$) was about 0.1 of the value of the first term ($k = 1$), and for $b/a = 5$ it was 0.05.

Figure 2 shows the temperature distribution in the mid-plane (when $Y = \pi/2$) of a square channel, while Fig. 1 shows the maximum temperature, calculated according to (9), at the mid-point of the channel as a function of the ratio of the sides. It follows from Fig. 1 that the dimensionless number $t\lambda/\mu\overline{W}_Z^2$ reaches a maximum when $b/a = 1$; with increase of the ratio of the sides the quantity $t\lambda/\mu\overline{W}_Z^2$ falls sharply, and in the limit, when $b/a \to \infty$, it reaches the value 0.75 given in [1] for the case of established laminar flow in a narrow plane channel.

NOTATION

W_Z is the stream velocity; x, y are the coordinates perpendicular to flow direction; z is the coordinate along flow; μ is the viscosity of liquid; λ is the thermal conductivity of liquid; t is the temperature of liquid; P is the pressure; $\bar{W}_Z$ is the mean mass flowrate; q is the heat flux at wall; $(\partial t/\partial n)_W$ is the temperature gradient at wall.

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